An Exposition of Cessi's Ocean Box Model

Kate Meyer

March 31, 2015

Outline

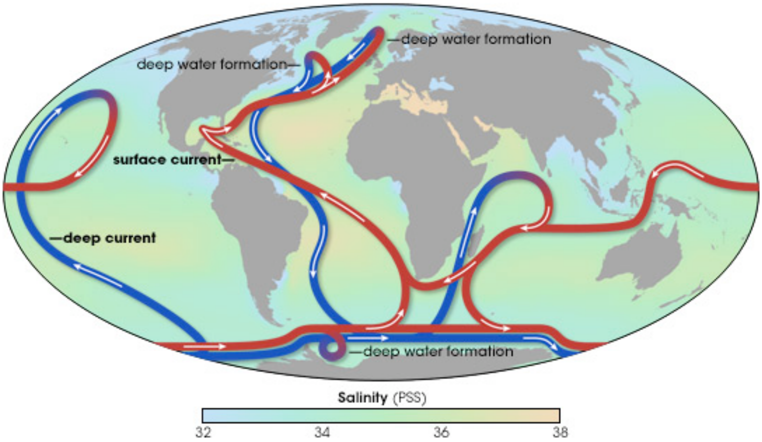
Motivations
- Understanding ocean circulation
- Modeling resilience to transient parameter changes

The box model
- Setup
- Dimension reduction

Deterministic perturbations

Conclusions and Future Directions
- Cessi's calculation
- Questions to pursue

Cessi's motivation: understanding ocean circulation



http://earthobservatory.nasa.gov/Features/Paleoclimatology_Evidence/paleoclimatology_evidence_2.php

Cessi's motivation: understanding ocean circulation

In models, competing influences of temperature and salinity give rise to alternative stable flow states:

e.g. Stommel (1961), Welander (1986), Manabe and Stouffer (1988)

Cessi's motivation: understanding ocean circulation

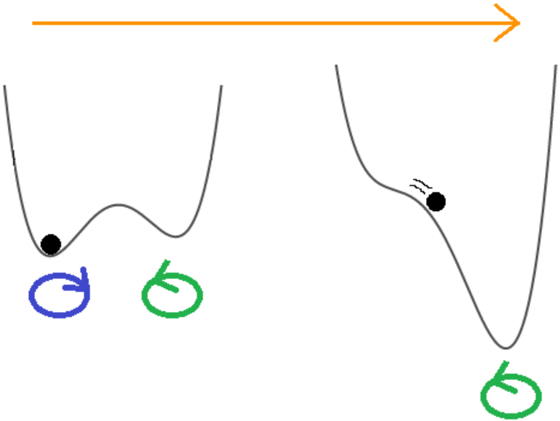
The Younger Dryas

A period of abnormally cold temperatures that occurred between 12,900 and 11,500 years BP during the last deglaciation

Broecker, Wallace S. (2006). "Was the Younger Dryas Triggered by a Flood?". Science 312 (5777): 11461148

Cessi's motivation: understanding ocean circulation

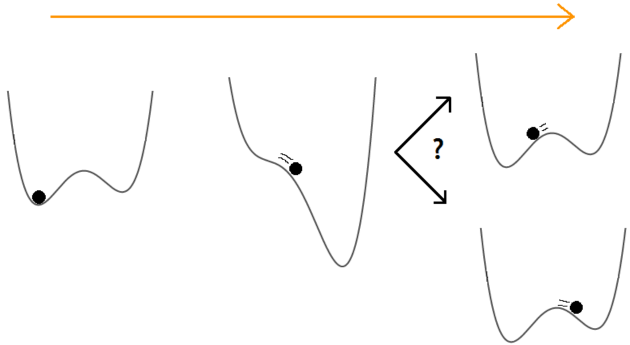
altered salinity forcing



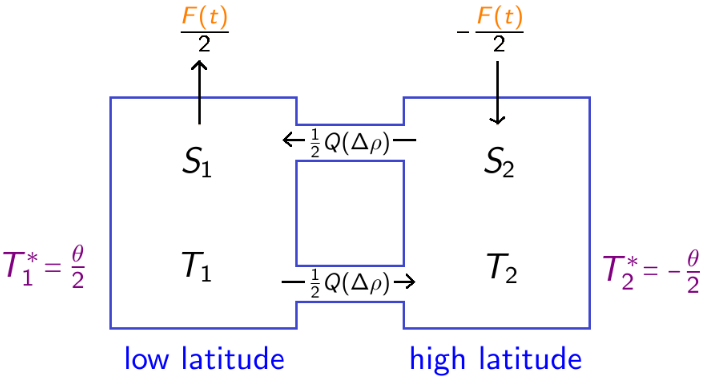
My motivation

Modeling resilience to

transient parameter changes

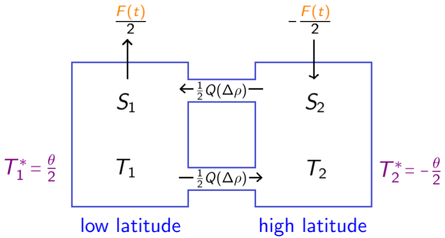


The box model: setup



$$\Delta\rho = \alpha_S(S_1 - S_2) - \alpha_T(T_1 - T_2)$$
$$Q(\Delta\rho) = \frac{1}{t_d} + \frac{q(\Delta\rho)^2}{V}$$

The box model: setup



$$T_1' = -\frac{1}{t_r}\left(T_1 - \frac{\theta}{2}\right) - \frac{1}{2}Q(\Delta\rho)(T_1 - T_2)$$

$$T_2' = -\frac{1}{t_r}\left(T_1 + \frac{\theta}{2}\right) - \frac{1}{2}Q(\Delta\rho)(T_2 - T_1)$$

$$S_1' = \frac{F(t)}{2H}S_0 - \frac{1}{2}Q(\Delta\rho)(S_1 - S_2)$$

$$S_2' = -\frac{F(t)}{2H}S_0 - \frac{1}{2}Q(\Delta\rho)(S_2 - S_1)$$

The box model: dimension reduction

$$T_1' = -\frac{1}{t_r}\left(T_1 - \frac{\theta}{2}\right) - \frac{1}{2}Q(\Delta\rho)(T_1 - T_2) \qquad S_1' = \frac{F(t)}{2H}S_0 - \frac{1}{2}Q(\Delta\rho)(S_1 - S_2)$$

$$T_2' = -\frac{1}{t_r}\left(T_1 + \frac{\theta}{2}\right) - \frac{1}{2}Q(\Delta\rho)(T_2 - T_1) \qquad S_2' = -\frac{F(t)}{2H}S_0 - \frac{1}{2}Q(\Delta\rho)(S_2 - S_1)$$

$$\Delta T \equiv T_1 - T_2; \qquad \Delta S \equiv S_1 - S_2$$

$$\Delta T' = -\frac{1}{t_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$

$$\Delta S' = \frac{F(t)}{H}S_0 - Q(\Delta\rho)\Delta S$$

The box model: dimension reduction

$$\Delta T' = -\frac{1}{t_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$

$$\Delta S' = \frac{F(t)}{H} S_0 - Q(\Delta\rho)\Delta S$$

$$\Delta\rho = \alpha_S(S_1 - S_2) - \alpha_T(T_1 - T_2)$$

$$Q(\Delta\rho) = \frac{1}{t_d} + \frac{q(\Delta\rho)^2}{V}$$

$$x \equiv \frac{\Delta T}{\theta}, \quad y \equiv \frac{\alpha_S \Delta S}{\alpha_T \theta}, \quad t \equiv t_d t'$$

$$\alpha \equiv \frac{t_d}{t_r}, \quad p(t) \equiv \frac{\alpha_S S_0 t_d}{\alpha_T \theta H} F(t), \quad \mu^2 \equiv \frac{q t_d (\alpha_T \theta)^2}{V}$$

$$x' = -\alpha(x - 1) - x\left[1 + \mu^2 (x - y)^2\right]$$

$$y' = p(t) - y\left[1 + \mu^2 (x - y)^2\right]$$

The box model: dimension reduction

$$x' = -\alpha(x-1) - x\left[1+\mu^2(x-y)^2\right]$$
$$y' = p(t) - y\left[1+\mu^2(x-y)^2\right]$$

Since $\alpha \equiv \dfrac{t_d}{t_r}$ is very large,

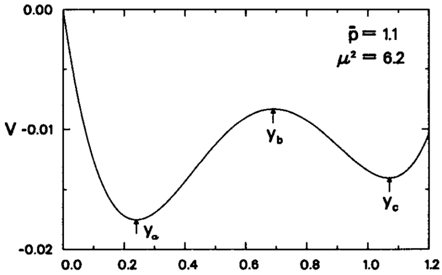
$$x = 1 + \mathcal{O}(\alpha^{-1})$$
$$y' = p(t) - y\left[1+\mu^2(1-y)^2\right] + \mathcal{O}(\alpha^{-1})$$

$$x \approx 1$$
$$y' \approx p(t) - y\left[1+\mu^2(1-y)^2\right]$$

The box model: one state variable

$$y' = p(t) - y\left[1 + \mu^2(1-y)^2\right]$$
$$= \bar{p} + \hat{p}(t) - y\left[1 + \mu^2(1-y)^2\right]$$

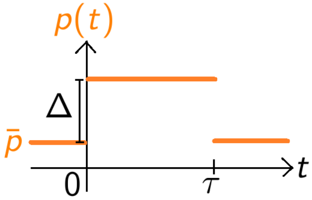
$$\hat{p}(t) = 0 \implies y' = -\frac{\partial}{\partial y}\Big[\overbrace{\mu^2\left(\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right) + \frac{y^2}{2} - \bar{p}y}^{V(y)}\Big]$$



Deterministic perturbations

$$y' = \bar{p} + \hat{p}(t) - y\left[1 + \mu^2(1-y)^2\right]$$

$$\text{Let } \hat{p}(t) = \begin{cases} 0 & t \leq 0 \\ \Delta & 0 \leq t \leq \tau \\ 0 & t > \tau \end{cases}$$



Deterministic perturbations

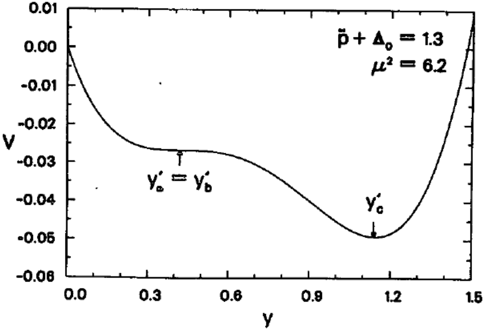


FIG. 4. The critical perturbation, of amplitude Δ_0, is found by requiring that the perturbed potential, shown here as a function of y, has one minimum and one inflection point. The perturbed potential is obtained by replacing $\bar{p}$ with $\bar{p} + \Delta$ in (2.9) and requiring that the global minimum y_a of Fig. 2 becomes the inflection point y'_a.

Deterministic perturbations

For $\Delta > \Delta_0$, how long can the system tolerate $p = \bar{p} + \Delta$ and still recover to y_a?

$$\frac{dy}{dt} = \bar{p} + \Delta - y\left[1 + \mu^2(1-y)^2\right]$$

$$\frac{dy}{\bar{p} + \Delta - y\left[1 + \mu^2(1-y)^2\right]} = dt$$

$$\int_{y_a}^{y_b} \frac{dy}{\bar{p} + \Delta - y\left[1 + \mu^2(1-y)^2\right]} = \int_0^{\tau} dt = \tau$$

[Why does this work?]

Deterministic perturbations

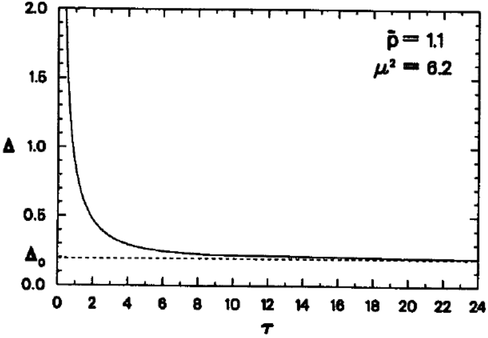


FIG. 3. The minimum amplitude of a perturbation, as a function of its duration, that will shift the system from the globally stable equilibrium y_a of Fig. 2 to the metastable state, y_c. The perturbation must exceed a critical amplitude, Δ_0, in order to displace the system from y_a, even if applied for an infinite time.

Cessi's calculation

- 1,000 years $\longrightarrow \tau = 4.6$
- Critical value of Δ for $\tau = 4.6$ is $\Delta \approx 0.3$
- $\Delta_0 = 0.2$ corresponds to freshwater flux of 0.4 $m\ yr^{-1}$
- Max meltwater flux preceeding Younger Dryas was 0.5 $m\ yr^{-1}$

"Close"

Questions to pursue

- What about $\hat{p}(t)$ continuous?
- How do $\mathcal{O}(\alpha^{-1})$ terms affect results?
- Extend calculations to higher dimensional systems?

Reference

Cessi, Paola (1994). A simple box model of stochastically forced thermohaline flow. *Journal of Physical Oceanography* v.24, pp. 1911-1920.